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On primitive Dirichlet characters and the Riemann hypothesis

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ABSTRACT

For every positive integer n, let $\mathcal{X}_n'$ be the set of primitive Dirichlet characters modulo n. We show that if the Riemann hypothesis is true, then the inequality $|\mathcal{X}_{2n_k}'| \leqslant C_2 e^{-\gamma} \varphi(2n_k)/\log\log(2n_k)$ holds for all $k \geqslant 1$, where n_k is the product of the first k primes, γ is the Euler–Mascheroni constant, C_2 is the twin prime constant, and $\varphi(n)$ is the Euler function. On the other hand, if the Riemann hypothesis is false, then there are infinitely many k for which the same inequality holds and infinitely many k for which it fails to hold.

1. Introduction

The purpose of this note is to exhibit for the first time a connection between the Riemann zeta function $\zeta(s)$ and the set of primitive Dirichlet characters χ. Our main result is a new reformulation of the classical Riemann hypothesis for $\zeta(s)$ in terms of collections of such characters. It is surprising that the twin prime constant makes an appearance in this context.

To be more precise, for every positive integer n let $\mathcal{X}_n$ be the set of Dirichlet characters modulo n and $\mathcal{X}_n'$ its subset of primitive characters. Let $\varphi(n)$ be the Euler function, γ the Euler–Mascheroni constant:

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log n\right) = 0.5772156649\cdots,$$

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and C_2 the twin prime constant:

$$C_2=\prod_{p>2}\frac{p(p-2)}{(p-1)^2}=0.6601618158\cdots.$$

Our result is the following:

Theorem 1. *For every $k\geqslant 1$, let n_k be the product of the first k primes.*

(i) *If the Riemann hypothesis is true, then the inequality*

$$\left|\mathcal{X}'_{2n_k}\right|\leqslant C_2e^{-\gamma}\frac{\varphi(2n_k)}{\log\log(2n_k)} \tag{1}$$

holds for all $k\geqslant 1$.

(ii) *If the Riemann hypothesis is false, then there are infinitely many k for which* (1) *holds and infinitely many k for which it fails to hold.*

Our work is motivated by and relies on the 1983 paper of J.-L. Nicolas [2] in which a relation is established between the Riemann hypothesis and certain values of the Euler function $\varphi(n)$; see also [3].

To prove Theorem 1, we introduce and study the ratios

$$\rho(n)=\frac{|\mathcal{X}'_n|}{|\mathcal{X}_n|}=\frac{|\mathcal{X}'_n|}{\varphi(n)}\quad(n\geqslant 1).$$

Note that the inequality (1) is equivalent to

$$\rho(2n_k)\log\log(2n_k)\leqslant C_2e^{-\gamma}. \tag{2}$$

For any natural number n, the value $\rho(n)$ is the proportion of Dirichlet characters modulo n that are primitive characters. Since $\rho(n)\leqslant 1$ for all $n\geqslant 1$, and $\rho(p)=1-1/(p-1)$ for every prime p, it is clear that

$$\limsup_{n\to\infty}\rho(n)=1.$$

As for the minimal order, we establish that

$$\liminf_{\substack{n\to\infty\\ n\not\equiv 2\ (\mathrm{mod}\ 4)}}\rho(n)\log\log n=C_2\,e^{-\gamma}. \tag{3}$$

Note that positive integers $n\equiv 2\pmod 4$ are excluded since $\rho(n)=0$ for those numbers; see (6) below. In Section 2, we show that the inequalities

$$2n_k\leqslant n\rho(2n_k)\leqslant\rho(n)\quad\bigl(n\not\equiv 2\ (\mathrm{mod}\ 4),\ \omega(n)=k>1\bigr) \tag{4}$$

hold, where $\omega(n)$ is the number of distinct prime divisors of n (for $k=1$ the same inequalities hold if $n>3$). Since (4) implies

$$\rho(2n_k)\log\log(2n_k)\leqslant\rho(n)\log\log n\quad\bigl(n\not\equiv 2\ (\mathrm{mod}\ 4),\ \omega(n)=k>1\bigr),$$

the assertion (3) is a consequence of

$$\lim_{k\to\infty} \rho(2n_k)\log\log(2n_k) = C_2 e^{-\gamma}, \tag{5}$$

which is also proved in Section 2.

In Section 3, we study the inequality (2) using techniques and results from [2], and these investigations lead to the statement of Theorem 1.

2. Small values of $\rho(n)$

Since $\rho(n) = |\mathcal{X}'_n|/\varphi(n)$ and

$$|\mathcal{X}'_n| = n\prod_{p\,\|\,n}\left(1-\frac{2}{p}\right)\prod_{p^2\mid n}\left(1-\frac{1}{p}\right)^2$$

(see, for example, [1, §9.1]), it follows that

$$\rho(n) = \frac{\varphi(n)}{n}\prod_{p\,\|\,n}\frac{p(p-2)}{(p-1)^2} \quad (n \geqslant 1). \tag{6}$$

Turning to the proof of (4), let $k > 1$ be fixed, and denote by $\mathcal{S}$ the set of integers $n \not\equiv 2 \pmod 4$ with $\omega(n) = k$. Let $p_1, p_2, \ldots$ be the sequence of consecutive prime numbers. For each integer $j \in \{0, \ldots, k\}$, let $\mathcal{S}_j$ be the set of numbers $n \in \mathcal{S}$ that have precisely j distinct prime divisors larger than p_k. Since $\mathcal{S}$ is the union of the sets $\mathcal{S}_j$, to prove (4) it suffices to show that the inequalities

$$2n_k \leqslant n\rho(2n_k) \leqslant \rho(n) \quad (n \in \mathcal{S}_j) \tag{7}$$

hold for every fixed $j \in \{0, \ldots, k\}$.

For every $n \in \mathcal{S}_0$ we can write $n = 2p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ with each $\alpha_j \geqslant 1$. Since $2n_k = 2p_1\cdots p_k$ it is clear that $2n_k \leqslant n$. Using (6) we also have

$$\rho(2n_k) = \rho(n)\prod_{\substack{j=2\\(\alpha_j\geqslant 2)}}^{k}\frac{p_j(p_j-2)}{(p_j-1)^2} \leqslant \rho(n),$$

which establishes (7) in the case that $j = 0$.

Proceeding by induction, let us suppose (7) has been established for some $j \in \{0, \ldots, k-1\}$. If n' is an arbitrary element of $\mathcal{S}_{j+1}$, then $q \mid n'$ for some prime $q > p_k$; note that $q \geqslant 5$ since $k > 1$. Writing $n' = q^\alpha m$ with $q \nmid m$, we have $\omega(m) = k-1$, hence for at least one index $i \in \{1, \ldots, k\}$ the prime p_i does not divide m. Put $n = p_i^\beta m$, where $\beta = 2$ if $p_i = 2$ and $\beta = 1$ otherwise. Clearly, $n \in \mathcal{S}_j$. Using (7) and the fact that $q > \max\{p_i, 2^2\}$, we have

$$2n_k \leqslant n = p_i^\beta m \leqslant q^\alpha m = n'.$$

In view of (6), we also have

$$\frac{\rho(n')}{\rho(m)} = \begin{cases} 1-1/(q-1) & \text{if } \alpha = 1, \\ 1-1/q & \text{if } \alpha \geqslant 2, \end{cases}$$

and

$$\frac{\rho(n)}{\rho(m)} = \begin{cases} 1 - 1/(p_i - 1) & \text{if } \beta = 1, \\ 1/2 & \text{if } \beta = 2. \end{cases}$$

Using (7) and the fact that $q > p_i$, it follows that $\rho(2n_k) \leqslant \rho(n) \leqslant \rho(n')$. Putting everything together, we have shown that

$$2n_k \leqslant n' \rho(2n_k) \leqslant \rho(n') \quad (n' \in \mathcal{S}_{j+1}),$$

which completes the induction and finishes our proof of (4).

Next, we turn to the proof of (5). Using the Prime Number Theorem in the form

$$\log n_k = \sum_{p \leqslant p_k} \log p = \bigl(1 + o(1)\bigr) p_k \quad (k \to \infty)$$

together with Mertens' theorem (see [1, Theorem 2.7(e)]), it is easy to see that

$$\lim_{k \to \infty} \left\{ \log\log(2n_k) \prod_{p \leqslant p_k} \left(1 - \frac{1}{p}\right) \right\} = e^{-\gamma}. \tag{8}$$

Also,

$$\lim_{k \to \infty} \prod_{2 < p \leqslant p_k} \frac{p(p-2)}{(p-1)^2} = \lim_{k \to \infty} C_2 \prod_{p > p_k} \left(1 + \frac{1}{p(p-2)}\right) = C_2. \tag{9}$$

By (6) one sees that

$$\rho(2n_k) = \prod_{p \leqslant p_k} \left(1 - \frac{1}{p}\right) \prod_{2 < p \leqslant p_k} \frac{p(p-2)}{(p-1)^2} \quad (k \geqslant 1),$$

and thus (5) is an immediate consequence of (8) and (9).

3. Proof of Theorem 1

As in [2, Théorème 3] we put

$$f(x) = e^{\gamma} \log \vartheta(x) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \quad (x \geqslant 2),$$

where $\vartheta(x) = \sum_{p \leqslant x} \log p$ is the Chebyshev ϑ-function. For our purposes, it is convenient to define

$$g(x) = e^{\gamma} \log\bigl(\vartheta(x) + \log 2\bigr) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \prod_{p > x} \left(1 + \frac{1}{p(p-2)}\right) \quad (x \geqslant 2).$$

This definition is motivated by the fact that

$$g(p_k) = C_2^{-1} e^{\gamma} \rho(2n_k) \log\log(2n_k) \quad (k \geqslant 1).$$

As mentioned earlier, the inequalities (1) and (2) are equivalent, and (2) is clearly equivalent to

$$\log g(p_k) \leqslant 0.$$

Thus, to prove Theorem 1 it suffices to study the sign of $\log g(x)$.

By the trivial inequality $\log(1+t) \leqslant t$ for all $t > -1$ and the fact that $g(x) > f(x)$ for all $x \geqslant 2$, it is easy to see that

$$0 < \log \frac{g(x)}{f(x)} \leqslant \frac{\log 2}{\vartheta(x)\log\vartheta(x)} + \frac{1}{x-2} \quad (x > 2). \tag{10}$$

Here, we have used the fact that

$$\sum_{p>x} \frac{1}{p(p-2)} \leqslant \sum_{n \geqslant \lfloor x \rfloor + 1} \frac{1}{n(n-2)} = \frac{1}{2}\left(\frac{1}{\lfloor x \rfloor} + \frac{1}{\lfloor x \rfloor - 1}\right) < \frac{1}{x-2} \quad (x > 2).$$

First, let us suppose that the Riemann hypothesis is true. In this case, we have from [2, p. 383]:

$$\log f(x) \leqslant -\frac{0.8}{\sqrt{x}\log x} \quad (x \geqslant 3000).$$

Using this bound in (10) together with the inequality $\vartheta(x) \geqslant 4x/5$ (which holds unconditionally for $x \geqslant 121$ by [4, Theorems 4 and 18]), one sees that

$$\log g(x) \leqslant \frac{\log 2}{(4x/5)\log(4x/5)} + \frac{1}{x-2} - \frac{0.8}{\sqrt{x}\log x} \leqslant -\frac{0.6}{\sqrt{x}\log x}$$

for all $x \geqslant 3000$. This implies the desired bound (2) for all $k \geqslant 431$; for smaller values of k, the bound (2) may be verified by a direct computation. This proves Theorem 1 under the Riemann hypothesis.

Next, suppose that the Riemann hypothesis is false, and let θ denote the supremum of the real parts of the zeros of the Riemann zeta function. Then, by [2, Théorème 3c] one has

$$\limsup_{x\to\infty} x^b \log f(x) > 0 \quad \text{and} \quad \liminf_{x\to\infty} x^b \log f(x) < 0$$

for any fixed number b such that $1-\theta < b < 1/2$. In view of (10) and the Chebyshev bound $\vartheta(x) \gg x$ it is clear that

$$\log g(x) = \log f(x) + O\left(x^{-1}\right);$$

hence, we also have

$$\limsup_{x\to\infty} x^b \log g(x) > 0 \quad \text{and} \quad \liminf_{x\to\infty} x^b \log g(x) < 0.$$

In particular, $\log g(p_k)$ changes sign infinitely often, which implies Theorem 1 if the Riemann hypothesis is false.

4. Remarks

Theorem 1 and its proof can be adapted to obtain additional reformulations of the Riemann hypothesis using a variety of arithmetical functions. Although it would be interesting to do so, we have not attempted to determine the most general conditions that lead to results of this type. In order to apply the results of J.-L. Nicolas [2] as we have done in this note, it is necessary to work with functions that are similar to $\varphi(n)/n$. Using the methods of Section 3 with the function $\exp(-\sum_{p\,|\,n} p^{-1})$, one can prove the following:

Theorem 2. *If the Riemann hypothesis is true, then the inequality*

$$\sum_{p \leqslant p_k} \frac{1}{p} \geqslant \log\log\log n_k + M \tag{11}$$

holds for all $k \geqslant 2$, where M is the Meissel–Mertens constant:

$$M = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = 0.2614972128\cdots.$$

If the Riemann hypothesis is false, then there are infinitely many k for which (11) *holds and infinitely many k for which it fails to hold.*

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Further reading

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